Two Proofs of the same Statement — ONE WITH STRONG Mathematical Induction and ONE Using the WELL-ORDERING PRINCIPLE

FIRST, THE PROOF USING STRONG MATHEMATICAL INDUCTION:

The sequence (a_n) is defined as follows: $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, and for all integers $k \geq 3$, $a_k = 3a_{k-2} + 2a_{k-3}$.

To Prove: For all integers $n \geq 0$, $a_n < 2^{n+1}$.

PROOF: [By STRONG MATH'L INDUCTION]

[BASIS STEP] Let $n = 0$. $\therefore a_n = a_0 = 1$ and $2^{n+1} = 2^{0+1} = 2$.

Since $1 < 2$, $a_n < 2^{n+1}$ for $n = 0$.

Let $n = 1$. $\therefore a_n = a_1 = 3$ and $2^{n+1} = 2^{1+1} = 4$.

Since $3 < 4$, $a_n < 2^{n+1}$ for $n = 1$.

Let $n = 2$. $\therefore a_n = a_2 = 5$ and $2^{n+1} = 2^{2+1} = 8$.

Since $5 < 8$, $a_n < 2^{n+1}$ for $n = 2$.

[END of BASIS STEP]

[INDUCTIVE STEP] Let k be any integer such that $k \geq 2$.

Suppose $a_m < 2^{m+1}$, for all integers m, $0 \leq m \leq k$. [Inductive Hypothesis]

[N.T.S: $a_{k+1} < 2^{(k+1)+1} = 2^{(k+2)}$]

Since $k \geq 2$, $k+1 \geq 3$

Since $k+1 \geq 3$, $a_{k+1} = 3a_{k-1} + 2a_{k-2}$ from the def'n of sequence (a_n).

Since $k \geq 2$, $k-2 \geq 0$. $\therefore 0 \leq k-2 < k-1 \leq k$.

Since $0 \leq k-2 \leq k$, $a_{k-2} < 2^{k-1}$ by the Inductive Hypothesis

Since $0 \leq k-1 \leq k$, $a_{k-1} < 2^{k}$ by the Inductive Hypothesis.

$\therefore 3a_{k-1} + 2a_{k-2} < (3 \times 2^{k} + 2 \times 2^{k-1})$

$\therefore a_{k+1} < (3 \times 2^{k} + 2 \times 2^{k-1})$ by substitution.

(EXAMPLE PROOF USING STRONG M.I.).

[From previous page: $\therefore a_{k+1} < (3\times 2^k + 2\times 2^{k-1})$]

By Rules of Algebra: $3\times 2^k + 2\times 2^{k-1} = 3\times 2^k + 1\times 2^k$

$= 4\times 2^k = 2^{k+2}$.

$\therefore (3\times 2^k + 2\times 2^{k-1}) = 2^{k+2}$.

$\therefore a_{k+1} < 2^{k+2}$ by substitution.

$\therefore$ For all integers $k \geq 2$, if $a_m < 2^{m+1}$ for all integers m such that $0 \leq m \leq k$, then $a_{k+1} < 2^{k+2}$, by Direct Proof.

[END of INDUCTIVE STEP].

$\therefore$ For all integers $n \geq 0$, $a_n < 2^{n+1}$, by the Principle of STRONG MATHEMATICAL INDUCTION.

Q.E.D.

NOW, A Proof of this statement
using the WELL-ORDERING PRINCIPLE.

The sequence (a_n) is defined as follows: $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, and for all integers $k \geq 3$, $a_k = 3a_{k-2} + 2a_{k-3}$.

To Prove: For all integers $n \geq 0$, $a_n < 2^{n+1}$.

Proof: [By proof-by-contradiction]

Suppose, by way of contradiction, that there exists an integer $N \geq 0$ such that $a_N \nless 2^{N+1}$.

Let set $S = \{$all integers t such that $t \geq 0$ & $a_t \nless 2^{t+1}\}$.

[Show that $S \neq \emptyset$] The integer $N \geq 0$ and $a_N \nless 2^{N+1}$.
$\therefore N \in S$. $\therefore S \neq \emptyset$.

[Show that the elements of S are all greater than or equal to a fixed integer.]

By definition of set S, every element in S is greater than or equal to 0.

$\therefore$ S satisfies the conditions of the Well-Ordering Principle of the Integers.

$\therefore$ By the Well-Ordering Principle, set S has a least element m.

(Example Proof using the Well-Ordering Principle (cont.))

INTERNAL LEMMA:

For all integers $t \geq 0$,
if $0 \leq t < m$, then $a_t < 2^{t+1}$.

Proof of the INTERNAL LEMMA:

Let t be any integer.
Suppose $0 \leq t < m$.
Since $t < m$ and m is the least element of set S,
t is NOT IN set S.
Suppose, BWOC, that $a_t \not< 2^{t+1}$; i.e. $a_t \geq 2^{t+1}$.

$\therefore$ t is in set S by definition of set S, which contradicts the fact that t is NOT IN set S.

$\therefore$ $a_t < 2^{t+1}$, by proof-by-contradiction.

Since $m \in S$, $m \geq 0$ and $a_m \nless 2^{m+1}$.

$a_0 = 1$ and $2^{0+1} = 2$. Since $1 < 2$, $a_0 < 2^{0+1}$. $\therefore m \neq 0$, since $a_m \nless 2^{m+1}$.	$a_1 = 3$ and $2^{1+1} = 4$. Since $3 < 4$, $a_1 < 2^{1+1}$. $\therefore m \neq 1$ since $a_m \nless 2^{m+1}$.	$a_2 = 5$ and $2^{2+1} = 8$. Since $5 < 8$, $a_2 < 2^{2+1}$. $\therefore m \neq 2$ since $a_m \nless 2^{m+1}$.

$\therefore m \geq 0$, but $m \neq 0$, $m \neq 1$, and $m \neq 2$. $\therefore m \geq 3$

Since $3 \leq m$, $0 \leq m-3 < m$

NOTE THAT $(m-3)+1 = m-2$

$\therefore$ By the Internal Lemma, $a_{m-3} < 2^{m-2}$.

Since $3 \leq m$, $0 \leq m-2 < m$, and note that $(m-2)+1 = m-1$.

$\therefore$ By the INTERNAL LEMMA, $a_{m-2} < 2^{m-1}$.

Since $m \geq 3$, $a_m = 3a_{m-2} + 2a_{m-3}$ by def'n of sequence (a_n).

Since $a_{m-2} < 2^{m-1}$ and $a_{m-3} < 2^{m-2}$, $3a_{m-2} + 2a_{m-3} < 3 \times 2^{m-1} + 2 \times 2^{m-2}$.

$\therefore a_m < 3 \times 2^{m-1} + 2 \times 2^{m-2}$ by substitution.

Now, $3 \times 2^{m-1} + 2 \times 2^{m-2} = 3 \times 2^{m-1} + 1 \times 2^{m-1}$

$= 4 \times 2^{m-1} = 2^{m+1}$.

$\therefore 3 \times 2^{m-1} + 2 \times 2^{m-2} = 2^{m+1}$

$\therefore a_m < 2^{m+1}$ by substitution.

$\therefore a_m < 2^{m+1}$ AND $a_m \nless 2^{m+1}$, which is a contradiction.

$\therefore$ For all integers $n \geq 0$, $a_n < 2^{n+1}$, by proof-by-contradiction.

QED.